# A NOTE ON PÓLYA'S OBSERVATION CONCERNING LIOUVILLE'S FUNCTION

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*Dedicated to Herman J. J. te Riele on the occasion of his retirement from the CWI in January 2012*

Abstract. We show that a certain weighted mean of the Liouville function $\lambda(n)$ is negative. In this sense, we can say that the Liouville function is negative "on average".

## 1. Introduction

For $n \in \mathbb{N}$ let $n = \prod_{p|n} p^{e_p(n)}$ be the canonical prime factorization of $n$ and let $\Omega(n) := \sum_{p|n} e_p(n)$. Here (as always in this paper) $p$ is prime. Thus, $\Omega(n)$ is the total number of prime factors of $n$, counting multiplicities. For example: $\Omega(1) = 0$, $\Omega(2) = 1$, $\Omega(4) = 2$, $\Omega(6) = 2$, $\Omega(8) = 3$, $\Omega(16) = 4$, $\Omega(60) = 4$, etc.

Define Liouville's multiplicative function $\lambda(n) = (-1)^{\Omega(n)}$. For example $\lambda(1) = 1$, $\lambda(2) = -1$, $\lambda(4) = 1$, etc. The Möbius function $\mu(n)$ may be defined to be $\lambda(n)$ if $n$ is square-free, and 0 otherwise.

It is well-known, and follows easily from the Euler product for the Riemann zeta-function $\zeta(s)$, that $\lambda(n)$ has the Dirichlet generating function

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}$$

for $\mathrm{Re}\,(s) > 1$. This provides an alternative definition of $\lambda(n)$.

Let $L(n) := \sum_{k\le n} \lambda(k)$ be the summatory function of the Liouville function; similarly $M(n) := \sum_{k\le n} \mu(k)$ for the Möbius function.

The topic of this note is closely related to Pólya's conjecture [12, 1919] that $L(n) \le 0$ for $n \ge 2$.

Pólya verified this for $n \le 1500$ and Lehmer [9, 1956] checked it for $n \le 600\,000$. However, Ingham [5, 1942] cast doubt on the plausibility of Pólya's conjecture by showing that it would imply not only the Riemann Hypothesis and simplicity of the zeros of $\zeta(s)$, but also the linear dependence over the rationals of the imaginary parts of the zeros

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$\rho$ of $\zeta(s)$ in the upper half-plane. Ingham cast similar doubt on the Mertens conjecture $|M(n)| \le \sqrt{n}$, which was subsequently disproved in a remarkable *tour de force* by Odlyzko and te Riele [11, 1985]. More recent results and improved bounds were given by Kotnik and te Riele [7, 2006]; see also Kotnik and van de Lune [6, 2004].

In view of Ingham's results, it was no surprise when Haselgrove showed [2, 1958] that Pólya's conjecture is false. He did not give an explicit counter-example, but his proof suggested that $L(u)$ might be positive in the vicinity of $u \approx 1.8474 \times 10^{361}$.

Sherman Lehman [8, 1960] gave an algorithm for calculating $L(n)$ similar to Meissel's [10, 1885] formula for the prime-counting function $\pi(x)$, and found the counter-example $L(906\,180\,359) = +1$.

Tanaka [14, 1980] found the smallest counter-example $L(n) = +1$ for $n = 906\,150\,257$. Walter M. Lioen and Jan van de Lune [*circa* 1994] scanned the range $n \le 2.5 \times 10^{11}$ using a fast sieve, but found no counter-examples beyond those of Tanaka. More recently, Borwein, Ferguson and Mossinghoff [1, 2008] showed that $L(n) = +1\,160\,327$ for $n = 351\,753\,358\,289\,465$.

Humphries [3, 4] showed that, under certain plausible but unproved hypotheses (including the Riemann Hypothesis), there is a limiting logarithmic distribution of $L(n)/\sqrt{n}$, and numerical computations show that the logarithmic density of the set $\{n \in \mathbb{N} | L(n) < 0\}$ is approximately 0.99988. Humphries' approach followed that of Rubinstein and Sarnak [13], who investigated "Chebyshev's bias" in prime "races".

Here we show in an elementary manner, and without any unproved hypotheses, that $\lambda(n)$ is (in a certain sense) "negative on average". To prove this, all that we need are some well-known facts about Mellin transforms, and the functional equation for the Jacobi theta function (which may be proved using Poisson summation). Our main result is:

**Theorem 1.** *There exists a positive constant $c$ such that for every (fixed) $N \in \mathbb{N}$*

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{e^{n\pi x} + 1} = -\frac{c}{\sqrt{x}} + \frac{1}{2} + O(x^N) \qquad \textit{as} \quad x \downarrow 0.$$

Thus, a weighted mean of $\{\lambda(n)\}$, with positive weights initially close to a constant (1/2) and becoming small for $n \gg 1/x$, is negative for $x < x_0$ and tends to $-\infty$ as $x \downarrow 0$.

In the final section we mention some easy results on the Möbius function $\mu(n)$ to contrast its behaviour with that of $\lambda(n)$.

## 2. Proof of Theorem 1

We prove Theorem 1 in three steps.

*Step 1.* For $x > 0$,

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{e^{n\pi x} - 1} = \phi(x) = \frac{\theta(x) - 1}{2},$$

where

$$\phi(x) := \sum_{k=1}^{\infty} e^{-k^2 \pi x}, \qquad \theta(x) := \sum_{k \in \mathbb{Z}} e^{-k^2 \pi x}.$$

*Step 2.* For $x > 0$,

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{e^{n\pi x} + 1} = \phi(x) - 2\phi(2x).$$

*Step 3.* Theorem 1 now follows from the functional equation

$$\theta(x) = \frac{1}{\sqrt{x}}\, \theta\left(\frac{1}{x}\right)$$

for the Jacobi theta function $\theta(x)$.

*Proof of Theorem 1.*

(1) In the following, we assume that $\mathrm{Re}\,(s) > 1$, so the Dirichlet series and integrals are absolutely convergent, and interchanging the orders of summation and integration is easy to justify.

As mentioned above, it is well-known that

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p \left(1 + p^{-s}\right)^{-1} = \prod_p \frac{1 - p^{-s}}{1 - p^{-2s}} = \frac{\zeta(2s)}{\zeta(s)}.$$

Define

$$f(x) := \sum_{n=1}^{\infty} \frac{\lambda(n)}{e^{nx} - 1}, \qquad (x > 0).$$

We will use the well known fact that if two sufficiently well-behaved functions (such as ours below) have the same Mellin transform then the functions are equal.

The Mellin transform of $f(x)$ is

$$\begin{aligned} F(s) &:= \int_0^\infty f(x)x^{s-1}\,\mathrm{d}x = \int_0^\infty \sum_{n=1}^\infty \frac{\lambda(n)}{e^{nx}-1}x^{s-1}\,\mathrm{d}x \\ &= \sum_{n=1}^\infty \lambda(n)\int_0^\infty \frac{x^{s-1}}{e^{nx}-1}\,\mathrm{d}x = \Big(\sum_{n=1}^\infty \frac{\lambda(n)}{n^s}\Big) \times \int_0^\infty \frac{x^{s-1}}{e^x-1}\,\mathrm{d}x \\ &= \frac{\zeta(2s)}{\zeta(s)} \times \zeta(s)\Gamma(s) = \zeta(2s)\Gamma(s). \end{aligned}$$

We also have

$$\begin{aligned} \int_0^\infty \phi\Big(\frac{x}{\pi}\Big)x^{s-1}\,\mathrm{d}x &= \int_0^\infty \Big(\sum_{n=1}^\infty e^{-n^2x}\Big)x^{s-1}\,\mathrm{d}x \\ &= \Big(\sum_{n=1}^\infty \frac{1}{n^{2s}}\Big) \times \int_0^\infty e^{-x}x^{s-1}\,\mathrm{d}x = \zeta(2s)\Gamma(s), \end{aligned}$$

so the Mellin transforms of $f(x)$ and of $\phi(x/\pi)$ are identical. Thus $f(x) = \phi(x/\pi)$. Replacing $x$ by $\pi x$, we see that

$$\sum_{n=1}^\infty \frac{\lambda(n)}{e^{n\pi x}-1} = \sum_{k=1}^\infty e^{-k^2\pi x},$$

completing the proof of step (1).

(2) Observe that

$$\frac{1}{e^{n\pi x}+1} = \frac{1}{e^{n\pi x}-1} - \frac{2}{e^{2n\pi x}-1},$$

so, from step (1),

$$\sum_{n=1}^\infty \frac{\lambda(n)}{e^{n\pi x}+1} = \phi(x) - 2\phi(2x).$$

(3) Using the functional equation for $\theta(x)$, we easily find that

$$\phi(x) - 2\phi(2x) = -\frac{c}{\sqrt{x}} + \frac{1}{2} + \frac{1}{\sqrt{x}}\Big(\phi\Big(\frac{1}{x}\Big) - \sqrt{2}\,\phi\Big(\frac{1}{2x}\Big)\Big)$$

with $c = (\sqrt{2}-1)/2 > 0$, proving our claim, since the "error" term is bounded by $\phi(1/x)/\sqrt{x} \sim \exp(-\pi/x)/\sqrt{x} = O(x^N)$ as $x \downarrow 0$ (for any fixed exponent $N$). $\square$

## 3. Remarks on the Möbius function

We give some further applications of the identity

$$(*) \qquad \frac{1}{z+1} = \frac{1}{z-1} - \frac{2}{z^2-1}$$

that we used (with $z = e^{n\pi x}$) in proving step (2) above.

**Lemma 2.** *For $|x| < 1$, we have*

$$\sum_{n=1}^{\infty} \mu(n) \frac{x^n}{x^n+1} = x - 2x^2.$$

*Proof.* Assume that $|x| < 1$. It is well known that

$$\sum_{n=1}^{\infty} \mu(n) \frac{x^n}{1-x^n} = x,$$

in fact this "Lambert series" identity is equivalent to the Dirichlet series identity $\sum \mu(n)/n^s = 1/\zeta(s)$. Writing $y = 1/x$, we have

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{y^n - 1} = 1/y.$$

If follows on taking $z = y^n$ in our identity $(*)$ that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{y^n+1} = \sum_{n=1}^{\infty} \frac{\mu(n)}{y^n-1} - 2\sum_{n=1}^{\infty} \frac{\mu(n)}{y^{2n}-1} = y^{-1} - 2y^{-2}.$$

Replacing $y$ by $1/x$ gives the result. □

**Corollary 3.**

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{2^n+1} = 0.$$

*Proof.* Take $x = 1/2$ in Lemma 2. □

If follows from Lemma 2 that

$$\lim_{x \uparrow 1} \sum_{n=1}^{\infty} \mu(n) \frac{x^n}{x^n+1} = -1,$$

so that one might say that in this sense $\mu(n)$ is negative on average. However, this is much weaker than what we showed in Theorem 1 for $L(n)$, where the corresponding sum tends to $-\infty$. The "complex-analytic" reason for this difference is that $\zeta(2s)/\zeta(s)$ has a pole (with negative residue) at $s = 1/2$, but $1/\zeta(s)$ is regular at $s = 1$.

## References

[1] P. Borwein, R. Ferguson and M. J. Mossinghoff, *Sign changes in sums of the Liouville function*, Math. Comp. **77** (2008), 1681–1694.

[2] C. B. Haselgrove, *A disproof of a conjecture of Pólya*, Mathematika **5** (1958), 141–145.

[3] P. B. Humphries, *The Summatory Function of Liouville's Function and Pólya's Conjecture*, Honours Thesis, Department of Mathematics, The Australian National University, Canberra, October 2010.

[4] P. B. Humphries, *The Distribution of Weighted Sums of the Liouville's Function and Pólya's Conjecture*, arXiv:1108.1524, August 2011.

[5] A. E. Ingham, *On two conjectures in the theory of numbers*, Amer. J. of Mathematics **64** (1942), 313–319.

[6] T. Kotnik and J. van de Lune *On the order of the Mertens function*, Experimental Mathematics **13**:4 (2004), 473–481.

[7] T. Kotnik and H. J. J. te Riele *The Mertens conjecture revisited*, Proc. ANTS 2006, Lecture Notes in Computer Science **4076** (2006), 156–167.

[8] R. S. Lehman, *On Liouville's function*, Math. Comp. **14** (1960), 311–320.

[9] D. H. Lehmer, *Extended computation of the Riemann zeta function*, Mathematika **3** (1956), 102–108.

[10] E. D. F. Meissel, *Berechnung der Menge von Primzahlen, welche innerhalb der ersten Milliarde naturlicher Zahlen vorkommen*, Math. Ann. **25** (1885), 251-257.

[11] A. M. Odlyzko and H. J. J. te Riele, *Disproof of the Mertens conjecture*, J. für die reine und angewandte Mathematik **357** (1985), 138–160.

[12] G. Pólya, *Verschiedene Bemerkungen zur Zahlentheorie*, Jahresbericht der deutschen Math.-Vereinigung **28** (1919), 31–40.

[13] M. Rubinstein and P. Sarnak, *Chebyshev's bias*, Experimental Mathematics **3** (1994), 173–197.

[14] M. Tanaka, *A numerical investigation on cumulative sum of the Liouville function*, Tokyo J. Math. **3** (1980), 187–189.

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